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ESTIMATION OF THE REACHABLE SET FOR THE PROBLEM OF VIBRATING KIRCHHOFF PLATE

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We consider a dynamical system with distributed parameters for the description of controlled vibrations of a Kirchhoff plate without polar moment of inertia. A class of optimal controls corresponding to finite-dimensional approximations is used to study the reachable set. Analytic estimates for the norm of these control functions are obtained depending on the boundary conditions. These estimates are used to study the reachable set for the infinite-dimensional system. For a model with incommensurable frequencies, an estimate of the reachable set is obtained under the condition of power decay of the amplitudes of generalized coordinates.

1. Introduction

The contemporary technical progress stimulates the development of new methods in the theory of optimal control over systems with distributed parameters. In particular, new algorithms for the control over spacecrafts must guarantee the stabilization not only of rigid structural elements but also of elastic elements [1]. For this reason, the problems of simulation and synthesis of the systems of control over elastic panels connected with solid bodies prove to be quite urgent [2].

In the study of vibrating plates, the Kirchhoff model is especially extensively used for the theoretical investigations [3–5]. A series of papers is devoted to the problems of control over the model of Kirchhoff plate. The bilinear problem of optimal control for the equation of the Kirchhoff plate fastened along a part of the boundary of the domain is studied in [6]. It is assumed that the distributed control force is proportional to the transverse component of the velocity of the plate at every point of the domain. The problem of active control with several time delays for the equation of free vibrations of a rectangular plate was considered in [7].

The many-dimensional linear vibrating systems of the block form were considered in [8]. A generalized method of modal control is presented for these systems, including the case where restrictions are imposed on the values of control. The problem of controllability for a class of linear infinite-dimensional systems with one-dimensional control was investigated in [9]. In the cited work, one can also find necessary and sufficient conditions for the exact, approximate, and zero-controllability of these systems.

The Kirchhoff model for a plate fastened to a rotating body was considered in [10, 11]. For this model, the authors studied a system of ordinary differential equations used for the description of vibrations with finitely many degrees of freedom.

The aim of the present paper is to investigate the reachable set of an infinite-dimensional dynamical system with incommensurable frequencies for control functions of a special form.

2. Description of the Model

A mathematical model of small vibrations of an elastic Kirchhoff plate hinged to the boundary of a body was proposed in [10]. The equations of motion of the analyzed problem for controlled rotations of the body about

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a fixed axis can be represented in the form

$$\dot{x}_{kj}(t) = A_{kj}x_{kj}(t) + B_{kj}u(t), \tag{1}$$

where

$$x_{kj}(t) = \begin{pmatrix} \xi_{kj}(t) \\ \eta_{kj}(t) \end{pmatrix}, \quad A_{kj} = \begin{pmatrix} 0 & \beta_{kj} \\ -\beta_{kj} & 0 \end{pmatrix}, \quad B_{kj} = \begin{pmatrix} 0 \\ \varphi_{kj} \end{pmatrix}, \quad u(t) \in \mathbb{R}, \quad \text{and} \quad (k,j) \in \mathbb{N}^2.$$

The quantities $\xi_{kj}(t)$ and $\eta_{kj}(t)$ are, respectively, the modal coordinate and velocity for the mode of vibrations with indices (k,j). The control $u(t)$ corresponds to the angular acceleration of the body (carrier).

The coefficients of Eqs. (1) are specified by the parameters of the elastic plate as follows:

$$\beta_{kj} = \alpha\left(\left(\frac{\pi k}{l_1}\right)^2 + \left(\frac{\pi j}{l_2}\right)^2\right),$$

$$\varphi_{kj} = \begin{cases} 0, & k \text{ is even}, \\ \dfrac{2l_2\sqrt{l_1 l_2}}{\pi^2 kj}, & k \text{ is odd}, \quad j \text{ is even}, \\ \dfrac{2\sqrt{l_1 l_2}(2a_2 - l_2)}{\pi^2 kj}, & k \text{ is odd}, \quad j \text{ is odd}, \end{cases} \qquad k, j \in \mathbb{N}^2.$$

Here, α, l_1, l_2, and a_2 are positive constants whose physical meaning is described in [10]. In what follows, we always assume that $2a_2 \neq l_2$.

We now introduce complex variables

$$z_{kj} = \xi_{kj} + i\eta_{kj},$$

$$\bar{z}_{kj} = \xi_{kj} - i\eta_{kj}.$$

Thus, system (1) in the variables z_{kj} and $\bar{z}_{kj}$ has the form

$$\begin{aligned} \dot{z}_{kj} &= -iz_{kj}\beta_{kj} + i\varphi_{kj}u(t), \\ \dot{\bar{z}}_{kj} &= i\bar{z}_{kj}\beta_{kj} - i\varphi_{kj}u(t). \end{aligned} \tag{2}$$

Since $\varphi_{kj} = 0$ for even indices k, system (1) has an uncontrolled subspace corresponding to the modes (ξ_{kj}, η_{kj}) with $(k,j) \in S$, where

$$S = \{(k,j) \in \mathbb{N}^2 \colon k \text{ is even}\}.$$

In what follows, we consider a subsystem of system (1) for the indices $\mathbb{N}^2 \setminus S$. Consider a given one-to-one mapping $n \longmapsto (k_n, j_n)$ according to which the index $n \in \mathbb{N}$ is associated with a pair of indices $(k_n, j_n) \in \mathbb{N}^2 \setminus S$.

Denote

$$\omega_n = \beta_{k_n j_n} = \alpha \left(\left(\frac{\pi k_n}{l_1} \right)^2 + \left(\frac{\pi j_n}{l_2} \right)^2 \right), \tag{3}$$

$$B_n = \varphi_{k_n j_n} = \begin{cases} \dfrac{2 l_2 \sqrt{l_1 l_2}}{\pi^2 k_n j_n}, & k_n \text{ is odd}, \quad j_n \text{ is even}, \\ \\ \dfrac{2 \sqrt{l_1 l_2}(2a_2 - l_2)}{\pi^2 k_n j_n}, & k_n \text{ is odd}, \quad j_n \text{ is odd}, \end{cases} \tag{4}$$

$$q_n = z_{k_n j_n}, \qquad q_{-n} = \bar{z}_{k_n j_n}.$$

This enables us to rewrite system (2) in the operator form as follows:

$$\dot{q} = Aq + Bu, \qquad q \in \ell^2, \quad u \in \mathbb{R}^1, \tag{5}$$

where

$$q = \begin{pmatrix} q_{-1} \\ q_1 \\ q_{-2} \\ q_2 \\ \vdots \end{pmatrix} \in \ell^2, \qquad A = i \begin{pmatrix} \omega_1 & 0 & 0 & 0 & \dots \\ 0 & -\omega_1 & 0 & 0 & \dots \\ 0 & 0 & \omega_2 & 0 & \dots \\ 0 & 0 & 0 & -\omega_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad B = i \begin{pmatrix} B_{-1} \\ B_1 \\ B_{-2} \\ B_2 \\ \vdots \end{pmatrix},$$

$$\omega_n = \beta_{k_n j_n}, \qquad B_n = \varphi_{k_n j_n}, \qquad B_{-n} = -\varphi_{k_n j_n}.$$

System (5) is considered in the Hilbert space ℓ^2 with the norm

$$\|q\|_{\ell^2} = \left(\sum_{n=1}^{\infty} \left(|q_n|^2 + |q_{-n}|^2 \right) \right)^{1/2}.$$

According to the Hille–Yosida theorem [12, p. 8], the operator $A\colon D(A) \longrightarrow \ell^2$ is an infinitesimal generator of a C_0-semigroup of linear operators $\{e^{tA}\}_{t \geq 0}$ in ℓ^2.

Hence, for any $q^0 \in \ell^2$, $\tau > 0$, and $u \in L^2(0, \tau)$, there exists a unique generalized solution $q(t, q^0, u)$ of Eq. (5) with $u = u(t)$, $t \in [0, \tau]$, satisfying the initial condition $q|_{t=0} = q^0$. This solution is given by the formula [12, p. 184]

$$q(t; q^0, u) = e^{tA} q^0 + \int_0^t e^{(t-s)A} B u(s) ds, \quad 0 \leq t \leq \tau.$$

Consider the reachable sets [13]:

$$R_\tau(q^0) = \{q^1 \in \ell^2 \colon q^1 = q(\tau; q^0, u) \text{ for } u \in L^2(0,\tau)\},$$

$$R(q^0) = \bigcup_{\tau \geq 0} R_\tau(q^0).$$

Recall that system (5) is approximately controlled if $\overline{R(q^0)} = \ell^2$ for all $q^0 \in \ell^2$. The Levan–Rigby criterion [14] is a standard method for the investigation of the controllability of systems of the form (5). This criterion reduces to the analysis of invariant subspaces of the adjoint semigroup $\{e^{tA^*}\}_{t \geq 0}$ in the kernel B^*. However, the direct application of this approach does not enable us to estimate the reachable sets $R_\tau(q^0)$ and to construct control functions guaranteeing the solution of two-point problems with given boundary conditions.

In the present paper, we use a family of functions corresponding to the finite-dimensional problems of optimal control in order to estimate the reachable set $R_\tau(q^0)$ of system (5).

Parallel with system (5), we consider its finite-dimensional subsystem corresponding to the coordinates q_{-n}, q_n, $n = \overline{1,N}$, for a fixed integer number $N \geq 1$:

$$\dot{\tilde{q}}_N = A_N \tilde{q}_N + B_N u, \tag{6}$$

$$A_N = i \begin{pmatrix} \omega_1 & 0 & \ldots & 0 & 0 \\ 0 & -\omega_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \omega_N & 0 \\ 0 & 0 & \ldots & 0 & -\omega_N \end{pmatrix}, \qquad \tilde{q}_N = \begin{pmatrix} q_{-1} \\ q_1 \\ \vdots \\ q_{-N} \\ q_N \end{pmatrix}, \qquad B_N = i \begin{pmatrix} B_{-1} \\ B_1 \\ \vdots \\ B_{-N} \\ B_N \end{pmatrix}.$$

The optimal control for a finite-dimensional system of the form (2) with quadratic quality functional was found in [11]. For system (6) with complex variables, the result obtained in [11] can be formulated as follows:

Lemma 1. *Let $\omega_j \neq \omega_k$ for all $1 \leq j \leq k \leq N$. Consider the problem of optimal control*

$$\dot{\tilde{q}}_N = A_N \tilde{q}_N + B_N u, \quad t \in [0,\tau], \tag{7}$$

$$J = \int_0^\tau |u(t)|^2 dt \longrightarrow \min, \tag{8}$$

$$\tilde{q}_N^0 = \tilde{q}_N(0) = \begin{pmatrix} q_{-1}^0 \\ q_1^0 \\ \vdots \\ q_{-N}^0 \\ q_N^0 \end{pmatrix} \in \mathbb{C}^{2N}, \qquad \tilde{q}_N^1 = \tilde{q}_N(\tau) = \begin{pmatrix} q_{-1}^1 \\ q_1^1 \\ \vdots \\ q_{-N}^1 \\ q_N^1 \end{pmatrix} \in \mathbb{C}^{2N}, \tag{9}$$

$$q_n^0 = \overline{q_{-n}^0}, \qquad q_n^1 = \overline{q_{-n}^1}, \qquad n = \overline{1, N}.$$

For the analyzed problem, the optimal control takes the form

$$\hat{u}_N(t) = \big(B_1 e^{i\omega_1 t}, B_{-1} e^{-i\omega_1 t}, \ldots, B_N e^{i\omega_N t}, B_{-N} e^{-i\omega_N t}\big)\nu,$$

where

$$\nu = \begin{pmatrix} \nu_{-1} \\ \nu_1 \\ \vdots \\ \nu_{-N} \\ \nu_N \end{pmatrix} = \begin{pmatrix} \frac{1}{B_1} & 0 & \ldots & 0 & 0 \\ 0 & \frac{1}{B_{-1}} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \frac{1}{B_N} & 0 \\ 0 & 0 & \ldots & 0 & \frac{1}{B_{-N}} \end{pmatrix} K^{-1} \begin{pmatrix} \frac{1}{B_{-1}} & 0 & \ldots & 0 & 0 \\ 0 & \frac{1}{B_1} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \frac{1}{B_{-N}} & 0 \\ 0 & 0 & \ldots & 0 & \frac{1}{B_N} \end{pmatrix} \big(e^{-i\omega_N \tau}\tilde{q}_N^1 - \tilde{q}_N^0\big), \tag{10}$$

$$K = (K_{jk})_{j,k=1}^N, \qquad K_{jj} = \begin{pmatrix} \tau & \frac{i\big(e^{-2i\omega_j\tau} - 1\big)}{2\omega_j} \\ \frac{i\big(1 - e^{2i\omega_j\tau}\big)}{2\omega_j} & \tau \end{pmatrix},$$

$$K_{jk} = i \begin{pmatrix} \frac{1 - e^{i(\omega_k - \omega_j)\tau}}{\omega_k - \omega_j} & \frac{e^{-i(\omega_k + \omega_j)\tau} - 1}{\omega_k + \omega_j} \\ \frac{1 - e^{i(\omega_k + \omega_j)\tau}}{\omega_k + \omega_j} & \frac{1 - e^{i(\omega_j - \omega_k)\tau}}{\omega_j - \omega_k} \end{pmatrix}, \quad j \neq k.$$

To study the reachable set of the infinite-dimensional system (5), we recall that a complex or real number χ is called an algebraic number if it is a root of a polynomial with integer coefficients which are not simultaneously equal to zero [15].

The number n^* is called a power of the algebraic number χ if χ is a root of a certain polynomial of degree n^* with integer coefficients and it is impossible to find a polynomial with integer coefficients of degree smaller than n^* which is not identically equal to zero and has the root χ.

We now formulate the main result of the present paper on the estimation of the states $q^1 \in \ell^2$ of system (5) approximately reachable from the point $q^0 = 0 \in \ell^2$.

Theorem 1. *Let the following conditions be satisfied for system (5):*

(i) $B_n \neq 0,\ n = \pm 1, \pm 2, \pm 3, \ldots;$

(ii) $\chi = \left(\frac{l_2}{l_1}\right)^2$ *is an algebraic number of the power* $n^* \geq 2$;

(iii) the coordinates of the vector $q^1 = \begin{pmatrix} q^1_{-1} \\ q^1_1 \\ q^1_{-2} \\ q^1_2 \\ \vdots \end{pmatrix} \in \ell^2$ *satisfy the conditions*

$$q^1_{-n} = \overline{q^1_n}, \qquad |q^1_n| = O\left(\frac{1}{n^\gamma}\right), \quad \gamma > \frac{3}{2}n^* + 1,$$

for all $n \in \mathbb{N}$.

Then, for any $\varepsilon > 0$, *there exist numbers* $\tau = \tau(\varepsilon) > 0$ *and* $N(\varepsilon) \geq 1$ *such that*

$$\left\| q(\tau; 0, \hat{u}_N) - q^1 \right\|_{\ell^2} < \varepsilon, \tag{11}$$

where $\hat{u}_N(t)$ *is an optimal control for problem (7)–(9) of the form*

$$\hat{u}_N(t) = \sum_{n=1}^{N} (B_n e^{i\omega_n t} \nu_{-n} + B_{-n} e^{-i\omega_n t} \nu_n), \quad t \in [0; \tau].$$

Proof. Under the conditions of the theorem, χ is an irrational number. Hence, representation (3) yields the property $\omega_j \neq \omega_k$ for all $j \neq k$. Following the idea of [16], we estimate quantity (11) for controls of the form $u = \hat{u}_N(t)$ in Lemma 1.

We introduce projectors $Q_N \colon \ell^2 \longrightarrow \ell^2$ and $P_N \colon \ell^2 \longrightarrow \ell^2$ as follows:

$$P_N \colon q = \begin{pmatrix} q_{-1} \\ q_1 \\ \vdots \\ q_{-N} \\ q_N \\ q_{-N-1} \\ q_{N+1} \\ \vdots \end{pmatrix} \longmapsto \begin{pmatrix} q_{-1} \\ q_1 \\ \vdots \\ q_{-N} \\ q_N \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \qquad Q_N = I - P_N.$$

Then

$$\left\| q(\tau; 0, \hat{u}_N) - q^1 \right\|_{\ell^2} = \left\| Q_N q(\tau; 0, \hat{u}_N) - Q_N q^1 \right\|$$
$$\leq \left\| Q_N q^1 \right\| + \left\| \int_0^\tau Q_N e^{(\tau - s)A} B \hat{u}_N(s) ds \right\|.$$

Since the operators e^{tA} and Q_N are commuting, by applying the Cauchy–Buniakowski inequality, we obtain

$$\left\|q(\tau;0,\hat{u}_N)-q^1\right\|_{\ell^2} \le \left\|Q_N q^1\right\| + \sup_{t\in[0,\tau]}\|e^{tA}\|\,\|Q_N B\|\int\limits_0^\tau |\hat{u}_N(s)|ds$$

$$\le \left\|Q_N q^1\right\| + \sqrt{\tau}\|Q_N B\|\sup_{t\in[0,\tau]}\|e^{tA}\|\,\|\hat{u}_N\|_{L^2(0,\tau)}.$$

It is clear that the norm of the operator $e^{tA}\colon \ell^2 \longrightarrow \ell^2$ is equal to 1.

To prove the theorem, we show that, for any $\varepsilon > 0$ and vector q^1, there exist numbers $\tau > 0$ and N such that

$$\left\|Q_N q^1\right\| < \frac{\varepsilon}{2}, \tag{12}$$

$$r_N = \tau\|Q_N B\|^2\|\hat{u}_N\|^2_{L^2(0;\tau)} < \frac{\varepsilon^2}{4}. \tag{13}$$

Inequality (12) follows from the fact that

$$\lim_{N\longrightarrow\infty}\left\|Q_N q^1\right\|^2 = \lim_{N\longrightarrow\infty}\sum_{n=N+1}^{\infty}\left(|q^1_{-n}|^2+|q^1_n|^2\right)=0$$

for any element $q^1\in\ell^2$.

To prove inequality (13), we determine the value of the norm of the optimal control:

$$\left\|\hat{u}_N(t)\right\|^2_{L^2(0,\tau)} = \int\limits_0^\tau \left|\hat{u}_N(t)\right|^2 dt = \int\limits_0^\tau \hat{u}_N(t)\overline{\hat{u}_N(t)}dt$$

$$= \int\limits_0^\tau \left(\sum_{n=1}^N B_n e^{i\omega_n t}\nu_{-n} + B_{-n}e^{-i\omega_n t}\nu_n\right)\left(\sum_{n'=1}^N \bar{B}_{n'}e^{i\omega_{n'}t}\bar{\nu}_{-n'} + \bar{B}_{-n'}e^{-i\omega_{n'}t}\bar{\nu}_{n'}\right)dt$$

$$= \sum_{n=1}^N B_n\bar{B}_n\nu_{-n}\bar{\nu}_{-n}\tau + \sum_{n=1}^N B_n\bar{B}_{-n}\nu_{-n}\bar{\nu}_n\frac{i(1-e^{2i\omega_n\tau})}{2\omega_n}$$

$$+ \sum_{n=1}^N B_{-n}\bar{B}_n\nu_n\bar{\nu}_{-n}\frac{i(e^{-2i\omega_n\tau}-1)}{2\omega_n} + \sum_{n=1}^N B_{-n}\bar{B}_{-n}\nu_n\bar{\nu}_n\tau$$

$$+ \sum_{n=1}^N\sum_{n'=1}^N B_n\bar{B}_{n'}\nu_{-n}\bar{\nu}_{-n'}\frac{i(1-e^{i(\omega_n-\omega_{n'})\tau})}{\omega_n-\omega_{n'}}$$

$$+ \sum_{n=1}^N\sum_{n'=1}^N B_n\bar{B}_{-n'}\nu_{-n}\bar{\nu}_{n'}\frac{i(1-e^{i(\omega_n+\omega_{n'})\tau})}{\omega_n+\omega_{n'}}$$

$$+\sum_{n=1}^{N}\sum_{n'=1}^{N} B_{-n}\bar{B}_{n'}\nu_n\bar{\nu}_{-n'}\frac{i(e^{-i(\omega_n+\omega_{n'})\tau}-1)}{\omega_n+\omega_{n'}}$$

$$+\sum_{n=1}^{N}\sum_{n'=1}^{N} B_{-n}\bar{B}_{-n'}\nu_n\bar{\nu}_{n'}\frac{i(e^{i(\omega_{n'}-\omega_n)\tau}-1)}{\omega_n-\omega_{n'}}.$$

By using the triangle inequality and the Hölder inequality, we estimate $\left\|\hat{u}_N(t)\right\|^2_{L^2(0,\tau)}$ as follows:

$$\left\|\hat{u}_N(t)\right\|^2_{L^2(0,\tau)} \le \sum_{n=1}^{N}|B_n|^2|\nu_n|^2\left(2|\tau|+\frac{2}{|\omega_n|}\right)$$

$$+\sum_{\substack{n,n'=1\\ n\ne n'}}^{N}|B_nB_{n'}\nu_n\nu_{n'}|\left(\frac{4}{|\omega_n-\omega_{n'}|}+\frac{4}{|\omega_n+\omega_{n'}|}\right)$$

$$\le \sum_{n=1}^{N}|B_n|^2|\nu_n|^2\left(2|\tau|+\max_{n,n'\le N}\frac{2}{|\omega_n|}\right)$$

$$+\sum_{\substack{n,n'=1\\ n\ne n'}}^{N}|B_nB_{n'}\nu_n\nu_{n'}|\left(\max_{n,n'\le N}\frac{4}{|\omega_n-\omega_{n'}|}+\max_{n,n'\le N}\frac{4}{|\omega_n+\omega_{n'}|}\right)$$

$$\le \sum_{n=1}^{N}|B_n|^2|\nu_n|^2\left(2|\tau|+\frac{2}{\min\limits_{n,n'\le N}|\omega_n|}\right)$$

$$+\sum_{\substack{n,n'=1\\ n\ne n'}}^{N}|B_nB_{n'}\nu_n\nu_{n'}|\left(\frac{4}{\min\limits_{n,n'\le N}|\omega_n-\omega_{n'}|}+\frac{4}{\min\limits_{n,n'\le N}|\omega_n+\omega_{n'}|}\right)$$

$$\le \max_{n,n'\le N}\left\{\left(2|\tau|+\frac{2}{\min\limits_{n,n'\le N}|\omega_n|}\right),\left(\frac{4}{\min\limits_{n,n'\le N}|\omega_n-\omega_{n'}|}+\frac{4}{\min\limits_{n,n'\le N}|\omega_n+\omega_{n'}|}\right)\right\}$$

$$\times\sum_{\substack{n,n'=1\\ n\ne n'}}^{N}\left|B_nB_{n'}\nu_n\nu_{n'}\right|$$

$$\le \max_{n,n'\le N}\left\{\left(2\tau+\frac{2}{\omega_1}\right),\left(\frac{4}{\min\limits_{n,n'\le N}|\omega_n-\omega_{n'}|}+\frac{4}{\omega_1+\omega_2}\right)\right\}\sum_{\substack{n,n'=1\\ n\ne n'}}^{N}\left|B_nB_{n'}\nu_n\nu_{n'}\right|$$

$$\leq \frac{4}{\min\limits_{n,n'\leq N} |\omega_n - \omega_{n'}|} \sum_{\substack{n,n'=1 \\ n\neq n'}}^{N} |B_n B_{n'} \nu_n \nu_{n'}|. \tag{14}$$

Introducing the notation

$$\tilde{\nu}_n = \nu_n B_{-n}, \qquad \tilde{\nu}_{-n} = \nu_{-n} B_n, \qquad \frac{2}{\min\limits_{n,n'\leq N} |\omega_n - \omega_{n'}|} = H(N),$$

we reduce estimate (14) to the form

$$\left\|\hat{u}_N(t)\right\|^2_{L^2(0,\tau)} \leq 2H(N) \sum_{\substack{n,n'=1 \\ n\neq n'}}^{N} |\tilde{\nu}_n \tilde{\nu}_{n'}| = 2H(N) \sum_{n'=1}^{N} \left(\sum_{n=1}^{N} |\tilde{\nu}_n||\tilde{\nu}_{n'}| \right)$$

$$\leq 2H(N) \sum_{n'=1}^{N} \left(\sqrt{\sum_{n=1}^{N} |\tilde{\nu}_{n'}|^2} \sqrt{\sum_{n=1}^{N} |\tilde{\nu}_n|^2} \right) = H(N)\sqrt{2N}\|\tilde{\nu}\|_2 \sum_{n'=1}^{N} |\tilde{\nu}_{n'}|$$

$$\leq H(N)\sqrt{2N}\|\tilde{\nu}\|_2 \sqrt{\sum_{n'=1}^{N} |\tilde{\nu}_{n'}|^2} \sqrt{\sum_{n'=1}^{N} 1^2} = H(N)N\|\tilde{\nu}\|_2^2, \tag{15}$$

where

$$\|\tilde{\nu}\|_2 = \sum_{n=1}^{N} \left(|\tilde{\nu}_n|^2 + |\tilde{\nu}_{-n}|^2\right)^{\frac{1}{2}}$$

is the Euclidean norm of the vector $\tilde{\nu}$.

By Lemma 1, we get

$$\tilde{\nu} = K^{-1} y, \tag{16}$$

where

$$\tilde{\nu} = \begin{pmatrix} \tilde{\nu}_{-1} \\ \tilde{\nu}_1 \\ \vdots \\ \tilde{\nu}_{-N} \\ \tilde{\nu}_N \end{pmatrix} \qquad \text{and} \qquad y = \begin{pmatrix} \dfrac{1}{B_{-1}} & 0 & \dots & 0 & 0 \\ 0 & \dfrac{1}{B_1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dfrac{1}{B_{-N}} & 0 \\ 0 & 0 & \dots & 0 & \dfrac{1}{B_N} \end{pmatrix} (e^{-i\omega_N \tau} \tilde{q}_N^1 - \tilde{q}_N^0).$$

We now estimate the norm of $\tilde{\nu}$. To this end, we represent the matrix K in the form $K = \tau I + C$ and consider C as a linear operator from the space $\mathbb{C}^{2N}$ with the norm $\|\cdot\|_1$ into the space $\mathbb{C}^{2N}$ with the norm $\|\cdot\|_\infty$:

$$C\colon \left(\mathbb{C}^{2N}, \|\cdot\|_1\right) \longrightarrow \left(\mathbb{C}^{2N}, \|\cdot\|_\infty\right),$$

where

$$\|y\|_1 = \sum_{n=1}^{N} (|y_n| + |y_{-n}|)$$

and $\|y\|_\infty = \max\limits_{1\le |n|\le N} |y_n|$.

We rewrite (16) in the form

$$K\tilde{\nu} = (\tau I + C)\tilde{\nu} = y, \qquad \tilde{\nu} = \frac{y}{\tau} - \frac{C\tilde{\nu}}{\tau}.$$

This yields

$$\|\tilde{\nu}\|_\infty \le \frac{1}{\tau}\Big(\|y\|_\infty + \|C\tilde{\nu}\|_\infty\Big) \le \frac{1}{\tau}\Big(\|y\|_\infty + \|C\|\|\tilde{\nu}\|_1\Big),$$

$$\|\tilde{\nu}\|_\infty \left(1 - \frac{\|C\|}{\tau}\right) \le \frac{1}{\tau}\|y\|_\infty,$$

$$\|\tilde{\nu}\|_\infty \le \frac{\|y\|_\infty}{\tau - \|C\|} \qquad \text{under condition that} \quad \|C\| < \tau. \tag{17}$$

We estimate $\|C\|$ as follows:

$$\|C\| \le \max_{1\le n\le N}\left\{\max_{1\le m\le N}|C_{nm}|\right\} = \max_{1\le n\le N}\left\{\left|\frac{i(e^{-2i\omega_n\tau}-1)}{2\omega_n}\right|, \left|\frac{i(1-e^{2i\omega_n\tau})}{2\omega_n}\right|,\right.$$

$$\left.\max_{1\le m\le N}\left\{\frac{2}{|\omega_m-\omega_n|}, \frac{2}{|\omega_m+\omega_n|}\right\}\right\} \le \max_{1\le n\le N}\left\{\frac{1}{|\omega_n|}, \max_{1\le m\le N}\left\{\frac{2}{|\omega_m-\omega_n|}, \frac{2}{|\omega_m+\omega_n|}\right\}\right\}. \tag{18}$$

In view of relation (3), we get

$$\omega_n = \beta_{j_n k_n} = \alpha\left(\left(\frac{\pi j_n}{l_1}\right)^2 + \left(\frac{\pi k_n}{l_2}\right)^2\right), \qquad \omega_m = \beta_{j_m k_m} = \alpha\left(\left(\frac{\pi j_m}{l_1}\right)^2 + \left(\frac{\pi k_m}{l_2}\right)^2\right).$$

In view of the restrictions $1 \le n \le N$ and $1 \le m \le N$, we obtain inequalities of the form $1 \le j_m, k_m, j_n, k_n \le M(N)$ for some integer $M(N)$,

$$M(N) = O(\sqrt{N}) \qquad \text{as} \quad N \longrightarrow \infty. \tag{19}$$

Thus, we can estimate expression (18) as follows:

$$\|C\| \le \max_{j_n,k_n\le M}\left\{\frac{1}{\beta_{j_nk_n}}, \max_{\substack{(j_m,k_m)\ne(j_n,k_n)\\ j_n,k_n\le M}}\left\{\frac{2}{|\beta_{j_nk_n}+\beta_{j_mk_m}|}, \frac{2}{|\beta_{j_nk_n}-\beta_{j_mk_m}|}\right\}\right\}$$

$$\le \max_{j_n,k_n\le M}\left\{\frac{1}{\min\limits_{j_n,k_n\le M}\beta_{j_nk_n}},\ \frac{2}{\min\limits_{\substack{(j_m,k_m)\ne(j_n,k_n)\\ j_n,k_n,j_m,k_m\le M}}|\beta_{j_nk_n}+\beta_{j_mk_m}|},\ \frac{2}{\min\limits_{\substack{(j_m,k_m)\ne(j_n,k_n)\\ j_n,k_n,j_m,k_m\le M}}|\beta_{j_nk_n}-\beta_{j_mk_m}|}\right\}.$$

Setting $j_n=p,\ k_n=c,\ j_m=m,\ k_m=s,$ and $\chi=\left(\frac{l_2}{l_1}\right)^2$, we get

$$\min_{\substack{(j_m,k_m)\ne(j_n,k_n)\\ j_n,k_n,j_m,k_m\le M}}|\beta_{j_nk_n}-\beta_{j_mk_m}|=\frac{\alpha\pi^2}{l_2^2}\min_{\substack{(p,c)\ne(m,s)\\ p,c,m,s\le M}}|\chi p^2+c^2-\chi m^2-s^2|.$$

Since χ is an irrational algebraic number of power $n^*\ge 2,$ by the Liouville theorem, we obtain

$$\frac{\alpha\pi^2}{l_2^2}\min_{\substack{(p,c)\ne(m,s)\\ p,c,m,s\le M}}|\chi p^2+c^2-\chi m^2-s^2|\ge\min_{\substack{p\ne m\\ p,m\le M}}\frac{\alpha\pi^2R}{l_2^2|p^2-m^2|^{n^*-1}}\ge\frac{\alpha\pi^2R}{l_2^2(M^2(N)-1)^{n^*-1}},$$

where R is a positive constant depending only on χ and expressed in the explicit form via the quantities conjugate to χ [15].

Hence,

$$\|C\|\le\frac{l_2^2(M^2(N)-1)^{n^*-1}}{\alpha\pi^2R}.$$

It follows from relations (15), (17), and (19) that

$$\big\|\hat u_N(t)\big\|^2_{L^2(0,\tau)}\le NH(N)\|y\|_2^2\qquad\text{for}\quad\tau>\frac{l_2^2\big(M^2(N)-1\big)^{n^*-1}}{\alpha\pi^2R}=O(N^{n^*-1}).\tag{20}$$

We now estimate the Euclidean norm of the vector y:

$$\|y\|_2^2=\sum_{n=1}^N\big(|y_{-n}|^2+|y_n|^2\big)=\sum_{n=1}^N\frac{|e^{-i\omega_n\tau}q_n^1-q_n^0|^2+|e^{i\omega_n\tau}q_{-n}^1-q_{-n}^0|^2}{|B_n|^2}$$

$$\le 2\sum_{n=1}^N\frac{|e^{-i\omega_n\tau}q_n^1|^2+|e^{i\omega_n\tau}q_{-n}^1|^2+|q_n^0|^2+|q_{-n}^0|^2}{|B_n|^2}$$

$$\le 2\sum_{n=1}^N\frac{|q_n^1|^2+|q_{-n}^1|^2+|q_n^0|^2+|q_{-n}^0|^2}{|B_n|^2}.$$

Substituting the obtained expression in the left-hand side of relation (13), we get

$$r_N\le\frac{4N\tau}{\min\limits_{\substack{1\le n<\\ <n'\le N}}|\omega_n-\omega_{n'}|}\sum_{n=1}^N\frac{|q_n^1|^2+|q_{-n}^1|^2+|q_n^0|^2+|q_{-n}^0|^2}{|B_n|^2}\sum_{n=N+1}^\infty\big(|B_{-n}|^2+|B_n|^2\big).\tag{21}$$

We define mappings $n \longmapsto (p,c)$ and $n' \longmapsto (k,s)$ that associate the indices $n \in \mathbb{N}$ and $n' \in \mathbb{N}$ with the pairs of indices (p,c) and (k,s). According to the introduced notation, we find

$$B_n = i\varphi_{pc}, \qquad \omega_n = \beta_{pc}, \qquad \omega_{n'} = \beta_{ks},$$

where

$$\beta_{pc} = \alpha\left(\left(\frac{\pi p}{l_1}\right)^2 + \left(\frac{\pi c}{l_2}\right)^2\right) \qquad \text{and} \qquad \beta_{ks} = \alpha\left(\left(\frac{\pi k}{l_1}\right)^2 + \left(\frac{\pi s}{l_2}\right)^2\right).$$

Thus, relation (21) takes the form

$$r_N \le \frac{8N\tau}{\min\limits_{\substack{(p,c)\ne(k,s)\\ p,c,k,s\le N}} |\beta_{pc} - \beta_{ks}|} \sum_{p,c=1}^{N} \frac{|q_n^1|^2 + |q_{-n}^1|^2 + |q_n^0|^2 + |q_{-n}^0|^2}{|i\varphi_{pc}|^2} \sum_{p,c=N+1}^{\infty} |i\varphi_{pc}|^2. \tag{22}$$

Let

$$q_n^0 = 0, \qquad q_{-n}^0 = 0, \qquad |q_n^1| = |q_{-n}^1| = O\left(\frac{1}{p^\gamma} + \frac{1}{c^\gamma}\right).$$

In view of notation (4) for φ_{pc}, we get the following representation from relation (22):

$$r_N = O\left(\frac{N\tau}{\min\limits_{\substack{(p,c)\ne(k,s)\\ p,c,k,s\le N}} \left|\frac{\alpha\pi^2}{(l_1 l_2)^2}\left(l_1^2(c^2 - s^2) + l_2^2(p^2 - k^2)\right)\right|} \sum_{p,c=1}^{N} \frac{(c^\gamma + p^\gamma)^2}{(pc)^{2\gamma-2}} \sum_{p,c=N+1}^{\infty} (pc)^{-2}\right) \tag{23}$$

as $N \longrightarrow \infty$.

We estimate the expression

$$\min_{\substack{(p,c)\ne(k,s)\\ p,c,k,s\le N}} \left|\frac{\alpha\pi^2}{(l_1 l_2)^2}\left(l_1^2(c^2 - s^2) + l_2^2(p^2 - k^2)\right)\right|.$$

Let $\chi = \left(\frac{l_2}{l_1}\right)^2 > 0$,

$$1 - N^2 \le c^2 - s^2 = (c-s)(c+s) = mq \le N^2 - 1,$$

$$1 - N^2 \le p^2 - k^2 = (p-k)(p+k) = m'q' \le N^2 - 1.$$

Thus, we get

$$\min_{\substack{(p,c)\ne(k,s)\\ p,c,k,s\le N}} \left|\frac{\alpha\pi^2}{(l_1 l_2)^2}\left(l_1^2(c^2 - s^2) + l_2^2(p^2 - k^2)\right)\right| = \frac{\alpha\pi^2}{l_2^2} \min_{\substack{|mq|\le N^2-1\\ |m'q'|\le N^2-1}} |mq + \chi m'q'|.$$

If χ is an irrational algebraic number with power $n^* \geq 2,$ then, by the Liouville theorem,

$$|mq + \chi m'q'| = |m'q'| \left| \chi + \frac{mq}{m'q'} \right| > \frac{C|m'q'|}{|m'q'|^{n^*}} = \frac{C}{|m'q'|^{n^*-1}},$$

where C is a positive constant that depends only on χ and can be expressed in the explicit form via the quantities conjugate to χ.

If $1 \leq c \leq N,\ 1 \leq s \leq N,\ 1 \leq p \leq N,$ and $1 \leq k \leq N,$ then

$$\inf_{\substack{(m,m')\neq(0,0)\\ 2\leq q\leq 2N\\ 2\leq q'\leq 2N}} |mq + \chi m'q'| > \inf \frac{C}{|m'q'|^{n^*-1}} = \frac{C}{\sup |m'q'|^{n^*-1}} = \frac{C}{(2N(N-1))^{n^*-1}}.$$

Hence,

$$\min_{\substack{1\leq(p,c)\leq\\ \leq(k,s)\leq N}} \left| \frac{\alpha\pi^2}{(l_1 l_2)^2} \left(l_1^2(c^2 - s^2) + l_2^2(p^2 - k^2) \right) \right| = \frac{\alpha\pi^2}{l_2^2} \frac{C}{(2N(N-1))^{n^*-1}}. \tag{24}$$

Substituting (24) in (23), we obtain

$$r_N = O\left(\tau N (2N(N-1))^{n^*-1} \sum_{p,c=1}^{N} (pc)^{2-2\gamma} (c^\gamma + p^\gamma)^2 \sum_{p,c=N+1}^{\infty} (pc)^{-2} \right). \tag{25}$$

In (25), we perform the equivalent change

$$r_N = O\left(\tau N^{2n^*-1} \sum_{p,c=1}^{N} (pc)^{2-2\gamma} (c^\gamma + p^\gamma)^2 \sum_{p,c=N+1}^{\infty} (pc)^{-2} \right). \tag{26}$$

By applying the integral criterion of comparison to the sum in (26), we obtain

$$\sum_{p,c=1}^{N} (pc)^{2-2\gamma} (c^\gamma + p^\gamma)^2 \leq \int_0^N p^{2-2\gamma} dp \int_1^{N+1} c^2 dc$$

$$+ 2 \int_0^N p^{2-\gamma} dp \int_0^N c^{2-\gamma} dc + \int_1^{N+1} p^2 dp \int_0^N c^{2-2\gamma} dc$$

$$= \frac{2N^{6-2\gamma}}{(3-\gamma)^2} + \frac{2N^{3-2\gamma}\left((N+1)^3 - 1\right)}{9 - 6\gamma}$$

for $\gamma \neq 3$ and $\gamma \neq \dfrac{3}{2}$. Moreover,

$$\sum_{p,c=N+1}^{\infty} (pc)^{-2} \leq \int_N^\infty dp \int_N^\infty (pc)^{-2} dc = \int_N^\infty p^{-2} dp \int_N^\infty c^{-2} dc = \frac{1}{N^2}.$$

Substituting the values of the integrals in (26), we find

$$r_N = O\left(\frac{\tau 2N^{2n^*-1}N^{6-2\gamma}}{N^2(3-\gamma)^2} + \frac{\tau 2N^{2n^*-1}N^{3-2\gamma}((N+1)^3-1)}{N^2(9-6\gamma)}\right).$$

In the last equality, we perform the equivalent transformations

$$r_N = O\left(\frac{\tau N^{2n^*-1}N^{6-2\gamma}}{N^2} + \frac{\tau N^{2n^*-1}N^{3-2\gamma}N^3}{N^2}\right),$$

$$r_N = O\left(\tau N^{2n^*-2\gamma+3}\right). \qquad (27)$$

The obtained representation is true for the values $\tau = O(N^{n^*-1})$ satisfying inequality (20). By using (27), we get the values of γ for which $r_N \longrightarrow 0$ as $N \longrightarrow \infty$:

$$\gamma > \frac{3}{2}n^* + 1.$$

This inequality yields property (13) for sufficiently large N.

Hence, Theorem 1 is proved.

Corollary. *Let the conditions of Theorem 1 be satisfied. Then* $q^1 \in \overline{R(0)}$.

3. Conclusions

The proof of the main result of the present paper contains the theoretical substantiation of the applicability of the method of model analysis to the estimation of the reachable set of an infinite-dimensional system that describes vibrations of rectangular Kirchhoff plates. The key assumption guaranteeing the smallness of the norm of solutions of a subsystem with high-frequency modes in using the family of controls $u = \hat{u}_N(t)$ is the condition of Theorem 1 on the algebraic nature of the number $\chi = l_2^2/l_1^2$, where l_1 and l_2 are the sizes of the plate.

It is of interest to study the possibilities of weakening the conditions of Theorem 1 for the description of approximate controllability of the Kirchhoff model.

The problem of existence of the limit functions for the constructed family of controls $\{\hat{u}_N(t)\}$ as $N \longrightarrow \infty$ is also of interest for the subsequent investigation of the problem of controllability in Hilbert spaces.

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