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Prime ends in the mapping theory on the Riemann surfaces

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Abstract. The criteria for continuous and homeomorphic extensions to the boundary of mappings with finite distortion between domains on the Riemann surfaces by prime ends by Carathéodory are proved.

Keywords. Prime ends, Riemann sufaces, mappings of finite distortion, boundary behavior, Sobolev classes.

1. Introduction

The theory of the boundary behavior in the prime ends for the mappings with finite distortion has been developed in [12] for the plane domains and in [15] for the spatial domains. The pointwise boundary behavior of the mappings with finite distortion in regular domains on Riemann surfaces was recently studied by us in [30] and [31]. Moreover, the problem was investigated for regular domains on the Riemann manifolds for $n \geq 3$, as well as in metric spaces, see, e.g., [1] and [34]. It is necessary to mention also that the theory of the boundary behavior of Sobolev's mappings has significant applications to the boundary-value problems for the Beltrami equations and for analogs of the Laplace equation in anisotropic and inhomogeneous media, see, e.g., [3, 8, 10, 11, 13, 14, 20, 23, 26] and relevant references therein.

For the basic definitions, notations, discussions, and historic comments in the mapping theory on the Riemann surfaces, see our previous papers [29–32].

2. Definition of the prime ends and preliminary remarks

We act similarly to Carathéodory [5] for the definition of the prime ends of domains on a Riemann surface $\mathbb{S}$, see Chapter 9 in [6]. First of all, we recall that a continuous mapping $\sigma : \mathbb{I} \to \mathbb{S}$, $\mathbb{I} = (0, 1)$, is called a **Jordan arc** in $\mathbb{S}$, if $\sigma(t_1) \neq \sigma(t_2)$ for $t_1 \neq t_2$. We also use the notations σ, $\overline{\sigma}$, and $\partial\sigma$ for $\sigma(\mathbb{I})$, $\overline{\sigma(\mathbb{I})}$, and $\overline{\sigma(\mathbb{I})} \setminus \sigma(\mathbb{I})$, correspondingly. A Jordan arc σ in a domain $D \subset \mathbb{S}$ is called a **cross-cut** of the domain D, if σ splits D, i.e., $D \setminus \sigma$ has more than one (connected) component, $\partial\sigma \subseteq \partial D$, and $\overline{\sigma}$ is a compact set in $\mathbb{S}$.

A sequence $\sigma_1, \ldots, \sigma_m, \ldots$ of cross-cuts of D is called a **chain** in D, if

(i) $\overline{\sigma_i} \cap \overline{\sigma_j} = \varnothing$ for each $i \neq j$, $i, j = 1, 2, \ldots$;

(ii) σ_m splits D into 2 domains, one of which contains σ_{m+1}, and another one σ_{m-1} for each $m > 1$;

(iii) $\delta(\sigma_m) \to 0$ as $m \to \infty$.

Here, $\delta(E) = \sup\limits_{p_1, p_2 \in \mathbb{S}} \delta(p_1, p_2)$ denotes the diameter of a set E in $\mathbb{S}$ with respect to an arbitrary metric δ in $\mathbb{S}$ agreed with its topology, see [29–31].

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By definition, a chain of cross-cuts σ_m generates a sequence of domains $d_m \subset D$ such that $d_1 \supset d_2 \supset \ldots \supset d_m \supset \ldots$ and $D \cap \partial\, d_m = \sigma_m$. Two chains of cross-cuts $\{\sigma_m\}$ and $\{\sigma'_k\}$ are called **equivalent**, if, for each $m = 1, 2, \ldots$, the domain d_m contains all domains d'_k, except for a finite number, and, for each $k = 1, 2, \ldots$, the domain d'_k contains all domains d_m, except for a finite number, too. A **prime end** P of the domain D is an equivalence class of chains of cross-cuts of D. Below, E_D stands for the collection of all prime ends of a domain D, and $\overline{D}_P = D \cup E_D$ is its completion by prime ends.

Next, we say that a sequence of points $p_l \in D$ is **convergent to a prime end** P of D, if, for a chain of cross-cuts $\{\sigma_m\}$ in P, for each $m = 1, 2, \ldots$, the domain d_m contains all points p_l, except for their finite collection. Further, we say that a sequence of prime ends P_l converges to a prime end P, if, for a chain of cross-cuts $\{\sigma_m\}$ in P, for each $m = 1, 2, \ldots$, the domain d_m contains chains of cross-cuts $\{\sigma'_k\}$ in all prime ends P_l, except for their finite collection.

Now, let D be a domain in the compactification $\overline{\mathbb{S}}$ of a Riemann surface $\mathbb{S}$ by Kerekjarto–Stoilow, see a discussion in [29–31]. Open neighborhoods of points in D are induced by the topology of $\overline{\mathbb{S}}$. A basis of neighborhoods of a prime end P of D can be defined in the following way. Let d be an arbitrary domain from a chain in P. Denote by d^* the union of d and all prime ends of D having some chains in d. Just all such d^* form a basis of open neighborhoods of the prime end P. The corresponding topology on $\overline{D}_P$ is called the **topology of prime ends**.

Let P be a prime end of D on a Riemann surface $\mathbb{S}$, let $\{\sigma_m\}$ and $\{\sigma'_m\}$ be two chains in P, and let d_m and d'_m be the domains corresponding to σ_m and σ'_m. Then

$$\bigcap_{m=1}^{\infty} \overline{d_m} \subseteq \bigcap_{m=1}^{\infty} \overline{d'_m} \subset \bigcap_{m=1}^{\infty} \overline{d_m} \ ,$$

and, thus,

$$\bigcap_{m=1}^{\infty} \overline{d_m} = \bigcap_{m=1}^{\infty} \overline{d'_m} \ ,$$

i.e. the set named by a **body of the prime end** P,

$$I(P) := \bigcap_{m=1}^{\infty} \overline{d_m}, \tag{2.1}$$

depends only on P, but not on the choice of a chain of cross-cuts $\{\sigma_m\}$ in P.

It is necessary to note also that, for any chain $\{\sigma_m\}$ in the prime end P,

$$\Omega \ := \ \bigcap_{m=1}^{\infty} d_m \ = \ \varnothing \ . \tag{2.2}$$

Indeed, every point p in Ω belongs to D. Moreover, some open neighborhood of p in D should belong to Ω. In the contrary case, each neighborhood of p should have a point in some σ_m. However, in view of condition (iii), then $p \in \partial D$ that should contradict the inclusion $p \in D$. Thus, Ω is an open set and if Ω would be not empty, then the connectedness of D would be broken, because $D = \Omega \cup \Omega^*$ with the open set $\Omega^* := D \setminus I(P)$.

In view of conditions (i) and (ii), we have by (2.2) that

$$I(P) = \bigcap_{m=1}^{\infty} (\partial d_m \cap \partial D) = \partial D \ \cap \ \bigcap_{m=1}^{\infty} \partial d_m \ .$$

Thus, we obtain the following statement.